Four Multiplicative Cohomology Theorems

The main statement proved here is the

De Rham Theorem

The de Rham cohomology algebra of a paracompact manifold is canonically isomorphic to the graded algebra of endomorphisms of the constant sheaf viewed as an object in the derived category of sheaves of vector spaces.

There are similar statements for Lie algebra cohomology and group cohomology.

Reminder about the notion of derived category

Let $\mathcal{A}$ be an abelian category and $\mathcal{A}^\bullet$ the category of cochain complexes in $\mathcal{A}$. Recall that a morphism in $\mathcal{A}^\bullet$ is a **quasi-isomorphism** if it induces an isomorphism in cohomology. A **derived category** of $\mathcal{A}$ is a pair $(\mathcal{D}, i)$ where $\mathcal{D}$ is a category and i a functor from $\mathcal{A}^\bullet$ to $\mathcal{D}$ satisfying the following condition.

For any category $\mathcal{C}$ and any functor $F : \mathcal{A}^\bullet \to \mathcal{C}$ transforming quasi-isomorphisms into isomorphisms there is a unique functor $\widetilde{F} : \mathcal{D} \to \mathcal{C}$ such that

$$
\begin{array}{ccc}
\mathcal{A}^\bullet & \xrightarrow{\ F\ } & \mathcal{C} \\
{\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle \widetilde{F}} & \\
\mathcal{D} & &
\end{array}
$$

commutes.

1

It's easy to check that a derived category of $\mathcal{A}$ exists and is unique up to unique isomorphism, and that $\mathcal{D}$ has the same objects as $\mathcal{A}^\bullet$. It can be shown that $\mathcal{D}$ is additive.

For $C^\bullet \in \mathcal{A}^\bullet$ and $n \in \mathbb{Z}$ denote by $C^{n+\bullet}$ the indicated shifted complex equipped with $(-1)^n$ times the differential of $C^\bullet$. For $C^\bullet, D^\bullet \in \mathcal{A}^\bullet$ and $n \in \mathbb{Z}$ put

$$\operatorname{Ext}^n_\mathcal{A}(C^\bullet, D^\bullet) := \operatorname{Hom}_\mathcal{D}(C^\bullet, D^{n+\bullet})$$

and note that there is an obvious composition

$$\operatorname{Ext}^q_\mathcal{A}(D^\bullet, E^\bullet) \underset{\mathbb{Z}}{\otimes} \operatorname{Ext}^p_\mathcal{A}(C^\bullet, D^\bullet) \longrightarrow \operatorname{Ext}^{p+q}_\mathcal{A}(C^\bullet, E^\bullet),$$

called **cup-product**. Sometimes one abusively calls **derived category** the category $\mathcal{E}$ whose objects are those of $\mathcal{A}^\bullet$ and whose morphisms are defined by the rule

$$\operatorname{Hom}_\mathcal{E}(C^\bullet, D^\bullet) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^n_\mathcal{A}(C^\bullet, D^\bullet).$$

This short text is about the four following cohomology theories : the de Rham cohomology, the (relative) Lie algebra cohomology, the group cohomology and the Čech cohmology (I refer to Warner [W], Borel-Wallach [BW] and Cartan-Eilenberg [CE] for precise definitions, and to Verdier [V] for derived category theory). Each of these theories was first defined by a magic formula and then interpreted as an Ext-group ; in each case there is an obvious cup-product suggested by the formula and an obvious cup-product suggested by the Ext interpretation ; the goal is to show that the combinatorial cup-product is compatible with the conceptual one.

Statement of the Theorems

Let M be a paracompact manifold, $\mathbb{C}_M$ the constant sheaf with fiber $\mathbb{C}$ over M and $\mathbb{C}_M \rightarrowtail \Omega$ the de Rham resolution.

Multiplicative de Rham Theorem. The de Rham cohomology of M is canonically isomorphic as a graded algebra to $\mathrm{Ext}^\bullet_{\mathbb{C}_M\text{-mod}}(\mathbb{C}_M, \mathbb{C}_M)$.

Let k be a field of characteristic 0, let $\mathfrak{k} \subset \mathfrak{g}$ be finite dimensional Lie algebras over k ; assume $\mathfrak{g}$ is $\mathfrak{k}$-semisimple. Given a $(\mathfrak{g}, \mathfrak{k})$-module (*i.e.* a $\mathfrak{k}$-semisimple $\mathfrak{g}$-module) V, denote the Chevalley-Eilenberg cohomology of $(\mathfrak{g}, \mathfrak{k})$ with values in V by $H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{k}; V)$ and put $H^\bullet(\mathfrak{g}, \mathfrak{k}; V) := \mathrm{Ext}^\bullet_{\mathfrak{g}, \mathfrak{k}}(k, V)$. By a Theorem of Hochschild there is a canonical isomorphism of graded vector spaces

$$\Phi_V : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{k}; V) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, \mathfrak{k}; V).$$

Consider the diagram

$$
\begin{array}{ccc}
H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{k}; V) \otimes H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{k}; k) & \xrightarrow[\sim]{\Phi_V \otimes \Phi_k} & H^\bullet(\mathfrak{g}, \mathfrak{k}; V) \otimes H^\bullet(\mathfrak{g}, \mathfrak{k}; k) \\
\downarrow & & \downarrow \\
H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{k}; V) & \xrightarrow[\Phi_V]{\sim} & H^\bullet(\mathfrak{g}, \mathfrak{k}; V),
\end{array}
$$

where the vertical arrows represent the cup-products.

Multiplicative Hochschild Theorem. This diagram commutes.

Let G be a group and k a commutative ring. Denote the Eilenberg-MacLane cohomology of G with values in the G-module V by $H^\bullet_{\mathrm{EM}}(G, V)$ and put $H^\bullet(G, V) :=$

$\mathrm{Ext}^\bullet_G(k, V)$. By a Theorem of Eilenberg-MacLane there is a canonical isomorphism of graded k-modules

$$\Phi_V : H^\bullet_{\mathrm{EM}}(G, V) \xrightarrow{\sim} H^\bullet(G, V).$$

Consider the diagram

$$
\begin{array}{ccc}
H^\bullet_{\mathrm{EM}}(G, V) \otimes H^\bullet_{\mathrm{EM}}(G, k) & \xrightarrow[\sim]{\Phi_V \otimes \Phi_k} & H^\bullet(G, V) \otimes H^\bullet(G, k) \\
\downarrow & & \downarrow \\
H^\bullet_{\mathrm{EM}}(G, V) & \xrightarrow[\Phi_V]{\sim} & H^\bullet(G, V),
\end{array}
$$

where the vertical arrows represent the cup-products.

Multiplicative Eilenberg-MacLane Theorem. This diagram commutes.

Given an be an open cover $\mathcal{U}$ of a topological space X, a commutative ring k and a sheaf S of modules over the constant sheaf $C := k_X$ there is — by a results I'll call Čech Theorem — a canonical morphism of graded k-modules

$$\Phi_S : \check{H}^\bullet(\mathcal{U}, S) \to H^\bullet(X, S).$$

Consider the diagram

$$
\begin{array}{ccc}
\check{H}^\bullet(\mathcal{U}, S) \otimes \check{H}^\bullet(\mathcal{U}, C) & \xrightarrow{\Phi_S \otimes \Phi_C} & H^\bullet(X, S) \otimes H^\bullet(X, C) \\
c \downarrow & & \downarrow t \\
\check{H}^\bullet(\mathcal{U}, S) & \xrightarrow{\Phi_S} & H^\bullet(X, S),
\end{array}
$$

where c denotes the Čech cup-product and t denotes the "true" cup-product.

Multiplicative Čech Theorem. This diagram commutes.

Here is a theorem that is not multiplicative at all ; I put it here because its proof uses Čech cohomology — and because I don't know where else to put it.

Bott-Tu Theorem. The Fréchet structure of the de Rham cohomology of a paracompact manifold is a topological invariant.

Proof. In Proposition 9.8 of [BT] Bott and Tu give an explicit isomorphism φ from the cohomology $\check{H}^\bullet(\mathcal{U}, \mathbb{C})$ of a countable good cover onto the de Rham cohomology $H^\bullet_{dR}(M, \mathbb{C})$. A look at their formula shows that φ is continuous. Since the target is Fréchet, the Open Mapping Theorem implies that φ is a Fréchet isomorphism. Therefore the Fréchet structure of $\check{H}^\bullet(\mathcal{U}, \mathbb{C})$ doesn't depend on the choice of $\mathcal{U}$, and, similarly, the Fréchet structure of $H^\bullet_{dR}(M, \mathbb{C})$ doesn't depend on the choice of the differential structure of M. This proves that the Fréchet structure in question depends only on the topology of M. **QED**

Statement of the Propositions

Let k be a commutative ring and $\mathcal{C}$ a k-category ; the convention

$$\otimes := \underset{k}{\otimes}$$

shall be in force to the end. The following abbreviations will come handy : if A and B are complexes in $\mathcal{C}$ then $\langle A, B \rangle$ shall be the complex whose underlying graded k-module is defined by

$$\langle A, B \rangle^n := \underset{p}{\oplus} \, \mathrm{Hom}(C^p, D^{n+p}),$$

the differential being given by $df = d \circ f - (-1)^n \, f \circ d$ for $f \in \langle A, B \rangle^n$, and $[A, B]$ shall be its cohomology :

$$[A, B] := H(\langle A, B \rangle),$$

recalling that $[A, B]^n$ is also the k-module of homotopy classes of complex morphisms from $A^\bullet$ to $B^{n+\bullet}$. Let

$$can_{AB} : [A, B] \to \text{Ext}(A, B)$$

be the natural morphism. If C is a third complex then we have the composition morphisms

$$\langle B, C \rangle \otimes \langle A, B \rangle \to \langle A, C \rangle,$$

$$[B, C] \otimes [A, B] \to [A, C],$$

$$\text{Ext}(B, C) \otimes \text{Ext}(A, B) \to \text{Ext}(A, C),$$

all abusively denoted by c and the last one being by definition the cup-product ; moreover the following diagram commutes

$$
\begin{array}{ccc}
[B, C] \otimes [A, B] & \xrightarrow{\ \ c\ \ } & [A, C] \\
\left\downarrow{\scriptstyle can_{BC} \otimes can_{AB}}\right. & & \left\downarrow{\scriptstyle can_{AC}}\right. \\
\text{Ext}(B, C) \otimes \text{Ext}(A, B) & \xrightarrow[\ \ c\ \ ]{} & \text{Ext}(A, C).
\end{array}
\qquad (1)
$$

For $i = 1, 2, 3$ let V_i be an object of $\mathcal{C}$ and $\varepsilon_i : V_i \rightarrowtail A_i$ a right resolution of V_i.

6

Recall that we have the commuting diagram of isomorphisms

$$
\begin{array}{ccc}
\mathrm{Ext}(A_i, V_j) & \xrightarrow{\ \mathrm{Ext}(\varepsilon_i, V_j)\ } & \mathrm{Ext}(V_i, V_j) \\
\Big\downarrow {\scriptstyle \mathrm{Ext}(A_i, \varepsilon_j)} & & \Big\downarrow {\scriptstyle \mathrm{Ext}(V_i, \varepsilon_j)} \\
\mathrm{Ext}(A_i, A_j) & \xrightarrow[\ \mathrm{Ext}(\varepsilon_i, A_j)\ ]{} & \mathrm{Ext}(V_i, A_j),
\end{array}
$$

which I'll implicitly use to identify these four k-modules whenever convenient. Also I'll often write $f\,g$ for $f \circ g$.

Assume we are given complex morphisms

$$
\varphi_{ij} : \langle V_i, A_j \rangle \to \langle A_i, A_j \rangle
$$

for $i < j$, and

$$
\mu : \langle V_2, A_3 \rangle \otimes \langle V_1, A_2 \rangle \to \langle V_1, A_3 \rangle
$$

subject to

(a) $\varphi_{ij}(f)\,\varepsilon_i = f$ for all f in $\langle V_i, A_j \rangle$ and

(b) the diagram

$$
\begin{array}{ccc}
\langle V_2, A_3 \rangle \otimes \langle V_1, A_2 \rangle & \xrightarrow{\ \mu\ } & \langle V_1, A_3 \rangle \\
\Big\downarrow {\scriptstyle \varphi_{23} \otimes \varphi_{12}} & & \Big\downarrow {\scriptstyle \varphi_{13}} \\
\langle A_2, A_3 \rangle \otimes \langle A_1, A_2 \rangle & \xrightarrow[\ c\ ]{} & \langle A_1, A_3 \rangle
\end{array}
$$

commutes up to homotopy.

Let $\mu_* : [V_2, A_3] \otimes [V_1, A_2] \to [V_1, A_3]$ be the morphism induced by μ. Denote both $can_{V_i A_j}$ and $can_{A_i A_j}$ by can_{ij} when convenient. — Consider the diagram

$$
\begin{array}{ccc}
[V_2, A_3] \otimes [V_1, A_2] & \xrightarrow{\ \mu_*\ } & [V_1, A_3] \\
\big\downarrow{\scriptstyle can_{23} \otimes can_{12}} & & \big\downarrow{\scriptstyle can_{13}} \\
\mathrm{Ext}(V_2, V_3) \otimes \mathrm{Ext}(V_1, V_2) & \xrightarrow[c]{\ \ \ } & \mathrm{Ext}(V_1, V_3).
\end{array}
$$

Proposition 1. This diagram commutes.

Weak Proposition 2. If $\mathcal{C}$ has enough injectives and $\mathrm{Ext}^p(V_i, A_j^q) = 0$ for $p > 0$, $q \geq 0$ then $can_{ij} : [V_i, A_j] \to \mathrm{Ext}(V_i, V_j)$ is an isomorphism.

Proof. This follows immediately from Grothendieck's Remark 3 after Theorem 2.4.1 in [G]. **QED**

Strong Proposition 2. If $\mathrm{Ext}^p(V_i, A_j^q) = 0$ for $p > 0$, $q \geq 0$ then

$$can_{ij} : [V_i, A_j] \to \mathrm{Ext}(V_i, V_j)$$

is an isomorphism.

8

Proposition 1 and Weak Proposition 2

imply the Theorems

Proof of the Multiplicative de Rham Theorem. Let $\mathbb{C}_M \rightarrowtail \Omega$ the de Rham resolution. Put $\Omega(M) := \langle \mathbb{C}_M, \Omega \rangle$ as usual, and $V_1 = V_2 = V_3 := \mathbb{C}_M$, $A_1 = A_2 = A_3 := \Omega$, and define $\varphi : \Omega(M) \to \langle \Omega, \Omega \rangle$ by $\varphi(\alpha)(\xi) = \alpha \wedge \xi$ (here α is a form and ξ a germ) and $\mu : \Omega(M) \otimes \Omega(M) \to \Omega(M)$ by $\mu(\alpha, \beta) := \alpha \wedge \beta$. **QED**

Proof of the Multiplicative Hochschild Theorem. Let $\mathcal{C}$ be the category of $(\mathfrak{g}, \mathfrak{k})$-modules. Note that if W is a complex in $\mathcal{C}$ then $\langle k, W \rangle$ is the subcomplex $W^{\mathfrak{g}}$ of invariants. Put $V_1 = V_2 := k$, $V_3 := V$,

$$A_i := \mathrm{Hom}_k \left(U(\mathfrak{g}) \underset{\mathfrak{k}}{\otimes} \bigwedge(\mathfrak{g}/\mathfrak{k}) \, , \, V_i \right)_{K\text{-finite}}$$

and define $\varphi : A_1^{\mathfrak{g}} \to \langle A_1, A_j \rangle$ by $\varphi(\alpha)(\xi) = \alpha \wedge \xi$ and $\mu : A_1^{\mathfrak{g}} \otimes A_1^{\mathfrak{g}} \to A_3^{\mathfrak{g}}$ by $\mu(\alpha, \beta) := \alpha \wedge \beta$. **QED**

Proof of the Multiplicative Eilenberg-MacLane Theorem. Let $\mathcal{C}$ be the category of Gk-modules. Note that if W is a complex in $\mathcal{C}$ then $\langle k, W \rangle$ is the subcomplex W^G of invariants. To any Gk-module W attach the injective resolution $I(W)$ whose definition can be briefly recalled as follows : $I^n(W)$ is the Gk-module of maps from $G \times \cdots \times G$ ($n+1$ factors) to W, and the coboundary is given by

$$(df)(g_0, \ldots, g_n) = \sum (-1)^i \, f(g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

Also remember that the (combinatorial) cup-product $I^p(k) \times I^q(W) \to I^{p+q}(W)$ is given by

$$(\alpha \cup \beta)(g_0, \ldots, g_{p+q}) = \alpha(g_0, \ldots, g_p) \, \beta(g_p, \ldots, g_{p+q}).$$

Put $V_1 = V_2 := k$, $V_3 := V$, $A_i := I(V_i)$ and define $\varphi : A_1^G \to \langle A_1, A_j \rangle$ by $\varphi(\alpha)(\xi) = \alpha \cup \xi$ and $\mu : A_1^G \otimes A_1^G \to A_3^G$ by $\mu(\alpha, \beta) := \alpha \cup \beta$. **QED**

Proof of the Multiplicative Čech Theorem. For any sheaf T of C-modules let

$$\Psi_T : \check{H}^\bullet(\mathcal{U}, T) \to H^\bullet(X, T)$$

be the canonical morphism of graded k-modules. Consider the diagram

$$
\begin{array}{ccc}
\check{H}^\bullet(X, S) \otimes \check{H}^\bullet(X, C) & \xrightarrow{\Psi_S \otimes \Phi_C} & H^\bullet(X, S) \otimes H^\bullet(X, C) \\
{\scriptstyle c}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\
\check{H}^\bullet(X, S) & \xrightarrow[\Phi_S]{} & H^\bullet(X, S),
\end{array}
$$

where c and g denote respectively the Čech and the Godement cup-product as defined in section II.6.6 of [God]. Since this diagram commutes by observation (c) on page 257 of [God], it suffices to check that

$$
\begin{array}{ccc}
H^\bullet(X, S) \otimes H^\bullet(X, C) & \xrightarrow{\;=\;} & H^\bullet(X, S) \otimes H^\bullet(X, C) \\
{\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle t} \\
H^\bullet(X, S) & \xrightarrow[=]{} & H^\bullet(X, S),
\end{array}
$$

where, remember, t is the "true" cup-product, commutes. Let $\varepsilon_C : C \to A$ and $\varepsilon_S : C \to B$ be Godement's canonical resolutions and let $\mu_A : A \otimes_C A \to A$,

$\mu_B : B \otimes_C A \to B$ be the maps defined at the bottom of page 256 of [God]. To apply Propostion 1 we need maps

$$\varphi : \langle C, A \rangle \to \langle A, A \rangle,$$

$$\psi : \langle C, B \rangle \to \langle A, B \rangle,$$

$$\mu : \langle C, B \rangle \underset{k}{\otimes} \langle C, A \rangle \to \langle C, A \rangle.$$

Let's define them by setting

$$\big(\varphi(\alpha)\big)(a) := \mu_A\big(\alpha(x) \otimes a\big),$$

$$\big(\psi(\alpha)\big)(a) := \mu_B\big(\alpha(x) \otimes a\big)$$

for all $x \in X$ and all $a \in A(x)$ [the stalk over x] and letting μ be the composition of $\langle C, \mu_B \rangle$ with the canonical map

$$\langle C, B \rangle \underset{k}{\otimes} \langle C, A \rangle \to \langle C, B \underset{C}{\otimes} A \rangle.$$

Section II.6.6 of [God] implies then the assumptions of Proposition 1 are fulfilled.

QED

Proof of Proposition 1

I first claim that the diagram

$$
\begin{array}{ccc}
[V_i, A_j] & \xrightarrow{\;\;H(\varphi)\;\;} & [A_i, A_j] \\
\Big\downarrow {\scriptstyle can_{V_i A_j}} & & \Big\downarrow {\scriptstyle can_{A_i A_j}} \\
\mathrm{Ext}(V_i, A_j) & \xleftarrow[\mathrm{Ext}(\varepsilon_i, A_j)]{\;\;\;\;\;\;\;\;} & \mathrm{Ext}(A_i, A_j),
\end{array}
$$

commutes. Indeed, the restriction $[\varepsilon_i, A_j] : [A_i, A_j] \to [V_i, A_j]$ satisfying

$[\varepsilon_i, A_j] \, H(\varphi) = \mathrm{Id}_{[V_i, A_j]}$ by assumption (a) and

$$ can_{V_i A_j} \; [\varepsilon_i, A_j] = \mathrm{Ext}(\varepsilon_i, A_j) \; can_{A_i A_j} \, , $$

we have

$$ can_{V_i A_j} = can_{V_i A_j} \; [\varepsilon_i, A_j] \, H(\varphi) = \mathrm{Ext}(\varepsilon_i, A_j) \; can_{A_i A_j} \, H(\varphi). $$

The top square of

$$
\begin{array}{ccc}
[V_2, A_3] \, \otimes \, [V_1, A_2] & \xrightarrow{\;\;\mu_*\;\;} & [V_1, A_3] \\[2ex]
\Big\downarrow {\scriptstyle H(\varphi_{23}) \otimes H(\varphi_{12})} & & \Big\downarrow {\scriptstyle H(\varphi_{13})} \\[2ex]
[A_2, A_3] \, \otimes \, [A_1, A_2] & \xrightarrow{\;\;c\;\;} & [A_1, A_3] \\[2ex]
\Big\downarrow {\scriptstyle can_{23} \otimes can_{12}} & & \Big\downarrow {\scriptstyle can_{13}} \\[2ex]
\mathrm{Ext}(V_2, V_3) \, \otimes \, \mathrm{Ext}(V_1, V_2) & \xrightarrow[c]{\;\;\;\;\;\;} & \mathrm{Ext}(V_1, V_3)
\end{array}
$$

commutes by assumption (b) ; the bottom square commutes because it is of the form (1) ; the vertical compositions are respectively $can_{23} \otimes can_{12}$ and can_{13} by the claim. **QED**

Proof of Strong Proposition 2

For any complex C let $C[n]$ be the complex $C^{n+\bullet}$; for any complex morphism $f : B \to C[n]$ denote by $[f] \in [B, C]^n$ its homotopy class and by $\widetilde{f}$ the corresponding element of $\mathrm{Ext}^n(B, C)$; any object of $\mathcal{C}$ shall be viewed as a complex in degree

12

zero. — Let V be an object and A a complex satisfying $A^n = 0$ for $n < 0$ and $H^n(A) = 0$ for $n > 0$; let $Z \subset A$ be the subcomplex of cocycles (in other words A is a right resolution of Z^0) ; set

$$F^n := \operatorname{Ext}^n(V, -).$$

By left exactness of F^0 the canonical morphism $H^0(F^0 A) \to F^0 Z^0$ is an isomorphism, proving Strong Proposition 2 in degree 0. For $p > 0$ the short exact sequence $Z^{p-1} \rightarrowtail A^{p-1} \twoheadrightarrow A^p$ gives birth to the long exact sequence

$$
\begin{array}{ccccccc}
 & & & & 0 & \longrightarrow & \\[2mm]
F^0 Z^{p-1} & \longrightarrow & F^0 A^{p-1} & \longrightarrow & F^0 Z^p & \xrightarrow{\ \delta_{p,0}\ } & \\[2mm]
F^1 Z^{p-1} & \longrightarrow & F^1 A^{p-1} & \longrightarrow & F^1 Z^p & \xrightarrow{\ \delta_{p,1}\ } & \\[2mm]
F^2 Z^{p-1} & \longrightarrow & F^2 A^{p-1} & \longrightarrow & \cdots & \cdots & \\[2mm]
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

(see [V,III.1.2.5,p.162]). Let $\overline{\delta}_{p,0} : H^p(F^0 A) \to F^1 Z^{p-1}$ be the monomorphism induced by $\delta_{p,0}$ and for $r \geq 0$ introduce the inclusion

$$i_r : Z^r \hookrightarrow A[r+1].$$

Fix a positive integer n. We want to interpret in terms of connecting morphisms the canonical morphism $H^n(F^0 A) \to \operatorname{Ext}^n(V, Z^0)$, which I prefer to think as the morphism

$$can : H^n(F^0 A) \to \operatorname{Ext}^n(V, A)$$

satisfying

$$can([x]) = \widetilde{i}_n \ \widetilde{x} \tag{2}$$

for all morphisms $x : V \to Z^n$. Define

$$\psi : H^n(F^0 A) \to \mathrm{Ext}^n(V, A)$$

by

$$\psi := \mathrm{Ext}(V, i_0) \ \delta_{1,n-1} \ \delta_{2,n-2} \ \cdots \ \delta_{n-1,1} \ \overline{\delta}_{n,0} \ .$$

Strong Proposition 2 follows easily from

Lemma. We have $can = (-1)^{n(n+1)/2} \ \psi$.

Proof. For $p > 0$ let ι be the inclusion of Z^{p-1} into A^{p-1} and C_p the complex

$$C_p^{-1} = Z^{p-1} \ \xrightarrow{\ -\iota\ } \ A^{p-1} = C_p^0 \ ,$$

and consider the morphisms

$$Z^p \ \xleftarrow{\ q_p\ } \ C_p \ \xrightarrow{\ f_p\ } \ Z^{p-1}[1]$$

defined by

$$
\begin{array}{ccc}
Z^p & \xleftarrow{\ d\ } & A^{p-1} \\
& & \Big\uparrow{\scriptstyle -\iota} \\
& & Z^{p-1} \ \xrightarrow{\ =\ } \ Z^{p-1}[1]
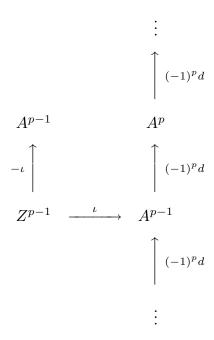\end{array}
$$

14

Note the following : $\widetilde{f}_p \in \operatorname{Ext}^1(C_p, Z^{p-1})$; $\widetilde{q}_p \in \operatorname{Ext}^0(C_p, Z^p)$; $\widetilde{q}_p$ is a quasi-isomorphism ; $\widetilde{f}_p \, \widetilde{q}_p^{-1} \in \operatorname{Ext}^1(Z^p, Z^{p-1})$. Recall that $\delta_{p,r} \; : \; F^r Z^p \;\to\; F^{r+1} Z^{p-1}$ coincides with the left multiplication by $\widetilde{f}_p \, \widetilde{q}_p^{-1}$ (see [Iv,XI.3]) ; in particular

$$\psi([x]) := \widetilde{i_0} \; \widetilde{f_1} \; \widetilde{q}_1^{-1} \; \widetilde{f_2} \; \widetilde{q}_2^{-1} \; \cdots \; \widetilde{f}_{n-1} \; \widetilde{q}_{n-1}^{-1} \; \widetilde{f}_n \; \widetilde{q}_n^{-1} \; \widetilde{x} \tag{3}$$
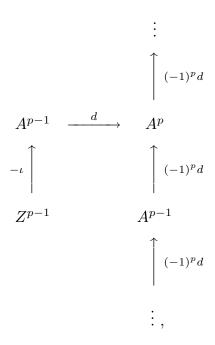
for $x \; : \; V \;\to\; Z^n$. Confronting (2) and (3) we see that the lemma reduces to the equality

$$\widetilde{i_0} \; \widetilde{f_1} \; \widetilde{q}_1^{-1} \; \widetilde{f_2} \; \widetilde{q}_2^{-1} \; \cdots \; \widetilde{f}_{n-1} \; \widetilde{q}_{n-1}^{-1} \; \widetilde{f}_n \; \widetilde{q}_n^{-1} = (-1)^n \; \widetilde{i}_n \, .$$

It suffices thus to check that for $p > 0$ we have $\widetilde{i}_{p-1} \; \widetilde{f}_p \; \widetilde{q}_p^{-1} \; = \; (-1)^p \; \widetilde{i}_p$, that is $\widetilde{i}_{p-1} \; \widetilde{f}_p \; = \; -\, \widetilde{i}_p \; \widetilde{q}_p$. The morphisms $i_{p-1} \, f_p$ and $i_p \, q_p$ from C_p to $A[p]$ being respectively given by the diagrams

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \Big\uparrow {\scriptstyle (-1)^p d} \\
A^{p-1} & & A^p \\
{\scriptstyle -\iota}\Big\uparrow & & \Big\uparrow {\scriptstyle (-1)^p d} \\
Z^{p-1} & \xrightarrow{\ \iota\ } & A^{p-1} \\
 & & \Big\uparrow {\scriptstyle (-1)^p d} \\
 & & \vdots
\end{array}
$$

and

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \big\uparrow \scriptstyle{(-1)^p d} \\
A^{p-1} & \xrightarrow{\ d\ } & A^p \\
{\scriptstyle -\iota}\big\uparrow & & \big\uparrow \scriptstyle{(-1)^p d} \\
Z^{p-1} & & A^{p-1} \\
 & & \big\uparrow \scriptstyle{(-1)^p d} \\
 & & \vdots \ ,
\end{array}
$$

the identity of A^{p-1} furnishes a homotopy from $i_{p-1}f_p$ to $(-1)^p\, i_p\, q_p$. **QED**

$$* \quad * \quad *$$

[BT] Bott R. & Tu Loring W., **Differential forms in algebraic topology**, Graduate Texts in Mathematics, 82, Springer-Verlag. XIV, 1982.

[BW] Borel A.& Wallach N., **Continuous cohomology, discrete subgroups, and representations of reductive groups**, Annals of Mathematics Studies, No. 94, Princeton University Press 1980.

[CE] Cartan H. & Eilenberg S., **Homological algebra**, Princeton University Press 1956.

[G] Grothendieck A., Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957) 119-221.

[God] Godement R., **Topologie algebrique et theorie des faisceaux**, Hermann, Paris 1958.

[Iv] Iversen B., **Cohomology of sheaves**, Springer 1986.

[V] Verdier, J.-L., **Des Catégories dérivées des catégories abéliennes**, Astérisque 239, SMF, Paris 1996.

[W] Warner, F., **Foundations of differentiable manifolds and Lie groups**, Springer-Verlag 1987.

Last update : Feb. 27, 2000

This text and others are available at http://www.iecn.u-nancy.fr/∼gaillard